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The localisation of energy in general relativity: II†

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Abstract. The energy localisation in the case of spherical symmetry advanced previously is justified in reference to an exact interior solution which is a generalisation of the $\rho =$ constant Schwarzschild solution. This reveals the inadequacy of the Tolman expression for localisation which also fails to provide the correct total mass for a system which exhibits certain discontinuities. A generalised localisation expression for arbitrary static systems is proposed for further consideration.

1. Introduction

The energy concept is central to physics and it is entirely natural that great efforts have been devoted through the years to examining its localisability (see, for example, Møller 1961, 1965). There is an understandable reluctance to resign oneself calmly to the generally prevalent idea that energy is globally conserved but that, in certain areas, one is not allowed to inquire about the routes of energy transport in dynamic situations nor, indeed, about its actual localisation in quiescent states. To resist this is in keeping with intuitive notions. Unless one can successfully elevate uncertainty to the level of a principle, as in quantum mechanics, it is best to strive first for certainty. Even if this should, in the end, prove unsuccessful, it would still be a positive step in emphasising the limitations of the concept.

Routes of gravitational energy transport have been considered on the Newtonian and relativistic levels (Bondi 1965, Cooperstock and Booth 1971, Synge 1972). Unfortunately, in spite of some interestingly suggestive expressions, there is a lack of uniqueness and one would be hard pressed to choose a preferred gravitational Poynting vector. In general relativity, integrated energy flux can be computed using one or other energy–momentum pseudotensor and the total mass loss confirmed in certain cases with the Bondi news function. However, the detailed transport is unknown.

The related and probably more basic question regarding the actual localisation of gravitational energy in static (and possibly also quasistatic) situations was considered by the authors and some preliminary observations were made (Cooperstock and Sarracino 1978). In this paper, we consider more fully the issues which have been raised and conclude that there are very sound arguments to support a particular localisation of

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energy for the case of spherical symmetry. This is reinforced by the analysis of an exact interior solution of the Einstein field equations which is a generalisation of the Schwarzschild (1916) interior solution $\rho=$ constant. The particular localisation assumption leads to a mass of the central core which is entirely consistent with that which would naturally be ascribed to it from the consideration of geodesics. Tolman (1930, 1962) found an expression for the mass of a static or quasistatic system which involves an integral over the stress–energy tensor and various authors have employed this to deduce the mass of a portion of the distribution, i.e. localisation of energy. However, we find that this expression, which we (Cooperstock and Sarracino 1978) had shown earlier to disagree with our preferred localisation, leads to an unacceptable measure of mass for the central core in the exact solution. Indeed, the Tolman expression fails to provide the correct total mass of the system as a whole, an unexpected result which is understood and rectified after a consideration of the role of the Gauss theorem in the derivation of the Tolman expression.

With spherical symmetry understood, we advance a proposal for more general static configurations to the effect that gravitational energy is most logically localised in regions of non-vanishing non-gravitational energy density. This is a straightforward resolution to the hitherto ambiguous situation regarding the localisation of gravitational field energy in vacuum: the energy is not there at all but rather tied to its source, matter. When a system becomes dynamic (with non-sphericity), this tie is broken and the gravitational energy leaves the source. Whether or not this process can in turn be localised is the object of future study.

2. Electromagnetic and gravitational energy

As a preliminary to the consideration of gravitational energy, it is helpful to consider the more tractable electromagnetic energy. Although one can concoct a variety of pairs of Poynting vectors and electromagnetic energy densities which satisfy a differential conservation law, the simplest and preferred pair is

$$\boldsymbol{S}=(c/4\pi)\,\boldsymbol{E}\times\boldsymbol{H} \qquad \rho_{\mathrm{EM}}=(E^2+H^2)/8\pi. \tag{2.1}$$

It has been suggested that the gravitational attraction of the assumed localisation would be a test of its validity (Feynmann *et al* 1964). At this point, it is worth indicating that such a test is readily constructed. Consider the Reissner–Nordström metric which describes the gravitational field of a spherically symmetric charge q with total mass m:

$$\mathrm{d}s^2=\left(1-\frac{2m}{r}+\frac{q^2}{r^2}\right)\mathrm{d}t^2-\left(1-\frac{2m}{r}+\frac{q^2}{r^2}\right)^{-1}\mathrm{d}r^2-r^2\,\mathrm{d}\Omega^2. \tag{2.2}$$

This metric is derived under the assumption that electromagnetic energy is distributed through space with non-vanishing energy–momentum tensor component $T_0^0=\rho_{\mathrm{EM}}$ of equation (2.1). Thus, a test of the geodesics for a spherically symmetric charge source constitutes a test of the assumed localisation according to whether or not they conform to the Reissner–Nordström metric.

It is interesting to note that the g_{00} component of the RN metric can be written as

$$1-\frac{2}{r}(m-q^2/2r)$$

and hence one can regard the effective mass which is perceived by a test particle at r as $m-q^2/2r$. The test particle senses the entire mass m as $r\rightarrow\infty$. For finite r, there is a shell of energy

$$\int_r^\infty \frac{E^2}{8\pi}\,\mathrm{d}V=\frac{q^2}{2r} \tag{2.3}$$

which is exterior (i.e. at larger r) to the test particle and hence does not drive its motion. It is also noteworthy that the integral in equation (2.3) which yields the mass deficit $q^2/2r$ is over coordinate volume, not proper volume. If it were taken over proper volume, it would be a measure of the electromagnetic energy in the shell. As it is, the integral measures total energy in the shell, including the gravitational contribution. This result is in line with the energy localisation expression which is discussed in the next section.

3. Avenue towards energy localisation

In earlier work, it was proposed that for a spherically symmetric system, the total energy up to radius r should be regarded as being localised as

$$m(r)=4\pi\int_0^r T_0^0 r'^2\,\mathrm{d}r' \tag{3.1}$$

in Schwarzschild coordinates. This was suggested by the fact that for a body whose energy–momentum tensor T_i^k extends to radius a, the total energy m is

$$m=4\pi\int_0^a T_0^0 r'^2\,\mathrm{d}r'. \tag{3.2}$$

There are at least three different attitudes which could be assumed with regard to localisation. The first is that while gravitational energy certainly contributes to the total mass of a system, its location is completely ambiguous, as exemplified by the ambiguities in the possible choices of energy–momentum pseudotensor for the gravitational field. This is probably the majority attitude among relativists who invoke the principle of equivalence as support. However, the essential ingredient of this support is unworthy of the title 'principle' according to Synge (1960), who suggests that it now be 'buried with appropriate honours'. Nevertheless, the attitude prevails and is expounded with vehemence by Misner *et al* (1973) who, curiously, couple it with the somewhat contradictory view that for spherically symmetric situations, energy is indeed localised according to equation (3.1). We refer to this amalgam as the second attitude which they justify primarily 'by the circumstance that transfer of energy (with this definition of m) is detectable by local measurements' and its exclusiveness to spherical symmetry rationalised because of the absence of spherical gravitational waves. Support for the localisation derives from the fact that

$$\dot{m}=-4\pi r^2 P\,\mathrm{d}r/\mathrm{d}t, \tag{3.3}$$

but is overstated because, while $4\pi r^2$ is the proper area of the surface at radius r, $\mathrm{d}r/\mathrm{d}t$ is not the proper velocity in the direction of the pressure forces. Hence, this statement is not what one might expect from an extension of the work–energy relationship to the realm of general relativity. However, it is a *bona fide* justification of the localisation in the weak-field limit and the form is certainly very suggestive.

The dichotomous approach toward localisation might appear somewhat illogical. If one were to grant the viability of the concept for spherical symmetry, it would at the very least appear premature to dismiss it for other cases merely because of the limitation of present knowledge with regard to the localisation of energy in gravitational radiation.

The third attitude which we will adopt is the following. If a global concept is meaningful and useful, these properties are enhanced if the concept can be carried over to the local level. The localisation of total energy including the gravitational contribution can be successfully formulated in the spherically symmetric case. Instead of asserting that this clearly defined localisation suddenly dissolves when one deviates from spherical symmetry, we attempt to extend it. In spherical symmetry, the localisation is tied to the T_i^k distribution according to equation (3.1). Perhaps this is also the case for all static distributions. When one attempts to distribute gravitational energy in the field, one encounters ambiguities which have influenced others to reject the concept entirely. By tying the energy to the T_i^k distribution, one removes these ambiguities. However, a problem which remains is that of describing the routes by which energy leaves the T_i^k distribution when a system becomes dynamic. If and when the localisation problem for static systems is thoroughly understood, this would be the next challenge.

In going beyond spherical symmetry, an obvious candidate for localisation which comes to mind was provided by Tolman (1930, 1962), who showed that for static or quasistatic systems with asymptotic Minkowskian boundary conditions, the total mass can be expressed as

$$m_{\mathrm{T}} = \int (T_0^0 - T_1^1 - T_2^2 - T_3^3)\sqrt{-g}\,\mathrm{d}V \tag{3.4}$$

where an integral over all space with the pseudotensor and Gauss' theorem has been used. A more direct derivation provided by Landau and Lifshitz (1975) bypasses the pseudotensor and derives from R_0^0:

$$\begin{aligned}\int R_0^0\sqrt{-g}\,\mathrm{d}V &= \int \frac{\partial}{\partial x^\alpha}(\sqrt{-g}g^{0i}\Gamma_{0i}^\alpha)\,\mathrm{d}V \\ &= \oint \sqrt{-g}g^{0i}\Gamma_{0i}^\alpha\,\mathrm{d}S_\alpha.\end{aligned} \tag{3.5}$$

The Tolman expression of equation (3.4) is then found by relating R_0^0 to the diagonal components of T_i^k via the field equations and the asymptotic field is used to evaluate the surface integral.

According to Tolman, the final expression in equation (3.4) has the 'great advantage that it can be evaluated by integrating only over the region actually occupied by matter or electromagnetic energy since the values of T_i^k will be zero in empty space.' Thus, for Tolman, the formula is apparently a convenient computational device for the evaluation of the total energy with no commitment to its localisation. However, through the years, various authors have used the formula to deduce energy for part of a system and have clearly misused it by applying it to systems of infinite extent, in violation of the asymptotic Minkowski constraint. We will demonstrate that not only is the localisation via the Tolman formula invalid, but also, more surprisingly, it does not always yield the correct total mass, even when the Tolman constraints are satisfied. This, and some of the other points which have been made, are developed by reference to an exact solution which is now considered.

4. Generalised interior Schwarzschild solution

A modification of the $T_0^0 =$ constant spherically symmetric solution found by Schwarzschild (1916) is readily found. This consists of an interior $\rho = \rho_0$ core (region I) of radius r_0 followed by a vacuum region II of outer radius r_1, which is surrounded by a $\rho = \rho_1$ shell (region III) of outer radius a. The outer vacuum region is labelled IV and ρ_0 and ρ_1 are both constants (see figure 1). The integration of the field equations

$$\begin{aligned} -8\pi\rho &= \mathrm{e}^{-\lambda}\left(\frac{1}{r^2}-\frac{\lambda'}{r}\right)-\frac{1}{r^2} \\ -8\pi P &= \frac{1}{r^2}-\mathrm{e}^{-\lambda}\left(\frac{1}{r^2}+\frac{\nu'}{r}\right) \\ -8\pi P &= \mathrm{e}^{-\lambda}\left(\tfrac{1}{4}\nu'\lambda'-\tfrac{1}{4}\nu'^2-\tfrac{1}{2}\nu''+\frac{(\lambda'-\nu')}{2r}\right) \end{aligned} \tag{4.1}$$

where $g_{00} = \mathrm{e}^{\nu}$, $g_{11} = -\mathrm{e}^{\lambda}$ is facilitated if the densities and coordinate radii satisfy the condition

$$\rho_0 r_0^3 = \rho_1 r_1^3. \tag{4.2}$$

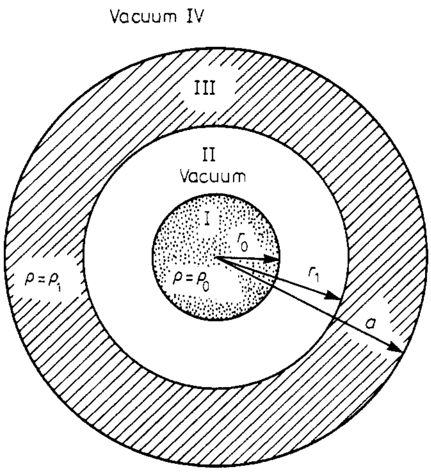

Figure 1. Generalised interior Schwarzschild model.

The solution in its entirety for the case of $P > 0$ is then

$$\text{I:} \qquad \mathrm{e}^{-\lambda} = 1-\tfrac{8}{3}\pi\rho_0 r^2 \qquad \mathrm{e}^{\nu} = \tfrac{1}{4}A(3B-\mathrm{e}^{-\lambda/2})^2$$

$$\rho = \rho_0 \qquad P = \rho_0\left(\frac{\mathrm{e}^{-\lambda/2}-B}{3B-\mathrm{e}^{-\lambda/2}}\right);$$

$$
\begin{aligned}
\text{II:}\quad & e^{-\lambda} = 1 - \frac{2m_0}{r} \qquad e^{\nu} = A\left(1 - \frac{2m_0}{r}\right) \\
& \rho = 0 \qquad P = 0; \\
\text{III:}\quad & e^{-\lambda} = 1 - \tfrac{8}{3}\pi\rho_1 r^2 \qquad e^{\nu} = \tfrac{1}{4}(3E - e^{-\lambda/2})^2 \\
& \rho = \rho_1 \qquad P = \rho_1\left(\frac{e^{-\lambda/2} - E}{3E - e^{-\lambda/2}}\right) \qquad E < e^{-\lambda/2} < 3E; \\
\text{IV:}\quad & e^{-\lambda} = e^{\nu} = 1 - \frac{2m}{r} \\
& \rho = 0 \qquad P = 0;
\end{aligned}
\tag{4.3}
$$

where

$$
\begin{aligned}
B^2 &\equiv 1 - \tfrac{8}{3}\pi\rho_0 r_0^{\,2} = 1 - \frac{2m_0}{r_0} \\
D^2 &\equiv 1 - \tfrac{8}{3}\pi\rho_1 r_1^2 = 1 - \frac{2m_0}{r_1} \\
E^2 &\equiv 1 - \tfrac{8}{3}\pi\rho_1 a^2 = 1 - \frac{2m}{a} \\
A &\equiv \left(\frac{3E}{2D} - \frac{1}{2}\right)^2.
\end{aligned}
\tag{4.4}
$$

Let us assume that the Tolman formula of equation (3.4) correctly localises the mass. Then the mass which is localised in the interior core I is easily found using equation (3.5) and integrating over a spherical surface in II. The result is

$$m_{\mathrm{T}}(\mathrm{I}) = \sqrt{A}\, m_0. \tag{4.5}$$

From equation (4.4), we note that this mass is dependent, through A, on the parameters of the exterior shell III. However, a study of the geodesic equations shows that a test particle in II is driven by m_0 and is completely uninfluenced by the spherically symmetric distribution which is exterior. This is the well known result of Newtonian theory which is also valid in general relativity. It is reasonable, then, to ascribe a mass m_0 to the core I and this is precisely what the localisation formula of equation (3.1) does. The Tolman formula fails here as it did in other cases studied earlier (Cooperstock and Sarracino 1978).

Let us now use the Tolman formula to compute the total mass of the system. Since all of the conditions demanded by Tolman are satisfied, it would be expected to render the correct answer. A direct calculation gives

$$m_{\mathrm{T}} = \tfrac{4}{3}\pi\rho_0\sqrt{A}\, r_0^3 + \tfrac{4}{3}\pi\rho_1(a^3 - r_1^3). \tag{4.6}$$

However, the total mass is unambiguously deduced from the asymptotic metric and, according to its form in equation (4.3), this is m. Moreover, from equations (4.2) and (4.4)

$$m = \tfrac{4}{3}\pi\rho_1 a^3 \tag{4.7}$$

and hence $m_{\mathrm{T}} \neq m$ unless $A = 1$, which implies the absence of the shell.

The discrepancy is resolved by noting that the Tolman expression, like many other formulae in physics, is derived with the aid of the Gauss theorem. A proper use of the theorem depends upon the careful examination of the continuity of relevant functions. Clearly, from equation (3.5), m_T will not express the total mass if there are discontinuities in $\sqrt{-g}g^{0i}\Gamma^{\alpha}_{0i}$. If such discontinuities exist, the total mass can still be found, but the total volume must be broken into regions of continuity and the Gauss theorem applied in each. This results in additional contributions to the mass which are given by the discontinuities in $\sqrt{-g}g^{0i}\Gamma^{\alpha}_{0i}$ over such surfaces. The particular solution given by equations (4.3) affords an excellent physically clear example.

For this spherically symmetric case,

$$\Gamma^2_{0i} = \Gamma^3_{0i} = 0$$

and hence the metric must be examined for discontinuities of

$$\sqrt{-g}g^{0i}\Gamma^1_{0i} = \tfrac{1}{2}r^2\nu' \, e^{(\nu-\lambda)/2}. \tag{4.8}$$

Now the shell (region III) is under pressure to maintain equilibrium and since the immediate inner region II is vacuum, there must be a sphere of support at $r = r_1$ to prevent collapse. Since T^k_i vanishes identically in II, the stress is zero for $r \to (r_1)_-$ and different from zero for $r \to (r_1)_+$. Thus, there is a discontinuity in P at $r = r_1$ and from the second of equations (4.1) and the continuity of the metric itself, ν' must be discontinuous at $r = r_1$. This is readily verified from equations (4.3), which give the following value for the discontinuity of the function in equation (4.8):

$$(\tfrac{1}{2}r^2\nu' \, e^{(\nu-\lambda)/2})_{r=r_1} = \tfrac{4}{3}\pi\rho_1 r_1^3(1-\sqrt{A}). \tag{4.9}$$

When this is added to the Tolman expression, equation (4.6), and the condition of equation (4.2) is used, the correct total mass $\frac{4}{3}\pi\rho_1 a^3$ results. The caution which is required in the use of the Gauss theorem has been referred to previously (Cooperstock and Hobill 1979). It is particularly satisfying to find such a clear-cut example.

While it is true that 'admissible' coordinate systems can be found in which the metric and its first derivatives are continuous, it is not necessarily a simple task and it is very often more convenient to work with other coordinate systems such as Schwarzschild coordinates in the example. As was pointed out by Schild (1967), λ' is discontinuous in the matter–vacuum interface for the $\rho = \text{constant}$ Schwarzschild solution but the present coordinates are conveniently used. The discontinuity in λ' also occurs in the present case in the I–II, II–III and III–IV interfaces, but from equation (4.8), this does not affect the mass calculation.

As in the Schwarzschild model, $P \to \infty$ at $r = 0$ if $r_0 = \frac{9}{8}(2m_0)$, and hence the radius of the inner core cannot attain this value while maintaining the static configuration. In the present model, there is an additional constraint which arises in that $P \to \infty$ at $r = r_1$ for

$$9\left(1-\frac{2m}{a}\right) = 1-\frac{2m_0}{r_1}.$$

This provides the general relativistic limit to the loading of the outer shell.

From the localisation expression of equation (3.1), one might conclude that the stresses do not contribute to the mass and hence the Tolman expression, which involves the stresses explicitly, is preferred. However, this would not be justified. Consider, for example, the $\rho = \rho_0 = \text{constant}$ static Schwarzschild sphere and a dust sphere with the same constant density and radius. They both have the same mass (Weinberg 1972).

The static sphere necessarily has stresses for support but the dust, by definition, is stress-free. The explanation is that although each has the same ρ distribution, it is differently constituted in each case. The net ρ function derives from rest mass, thermal energy and interaction energy, the last of which is the avenue by means of which the stresses contribute to ρ and hence to mass in the case of the static body.

5. Summary and a suggested generalisation

We have seen that there are a variety of reasons for accepting equation (3.1) as a reasonable expression for the localisation of energy in spherically symmetric matter distributions. While the Tolman expression of equation (3.4) appears attractive at first sight because it is not confined to spherical symmetry and because the density which it implies is a three-scalar (Møller 1952), it is inadequate. The expression does not localise in the manner of equation (3.1) when particularised to spherical symmetry and for the particular solution studied, it implied a mass for the inner core which is a function of the parameters of the outer shell, a physically unacceptable situation. Moreover, it was shown that the expression does not even necessarily yield the correct total mass when integrated over an entire body because its derivation employs Gauss' theorem which requires special care in the event that surface discontinuities exist. The exact solution chosen was precisely of this genre and the surface contributions in conjunction with the Tolman expression led to the correct total mass. These results underline the impropriety of using the Tolman expression, as one finds periodically in the literature, to deduce the partial mass of a system (and doubly so for an unbounded system which is clearly outside the scope of the derivation for the expression). The results also raise a new restriction on the applicability of the Tolman expression for deducing the total mass of a system.

A generalisation of the localisation for other static systems is now suggested. For any given static system, find a system of coordinates for which the total mass can be expressed as

$$m = \int T^0_0 \sqrt{{}^3g_F}\, d^3x$$

where ${}^3g_F$ is the determinant of the three-space metric in the limit $G \to 0$, i.e. the flat three-space metric base of the actual curved three-space. The total mass is known either from the asymptotic metric or from the Tolman expression (with due respect to its limitations). It is now suggested that the expression should serve as a measure of the localisation of the energy as well. It certainly works in the case of spherical symmetry from which it is, indeed, directly modelled. It generalises the property from the spherically symmetric case that although T^0_0 is not the complete energy density (the gravitational contribution is not included) and $\sqrt{{}^3g_F}\, d^3x$ is not the true volume element, the juxtaposition of the two in appropriate coordinate systems does account for the gravitational contribution. Assuming that this is the correct embodiment of the role of gravitation with regard to energy, the next challenge would be that of understanding how one finds the correct coordinates. In the case of spherical symmetry, the Schwarzschild coordinates are particularly suitable. Is it because the constant t, r two-surfaces are Euclidean in Schwarzschild coordinates? Beyond this lies the challenge of time-dependent systems.

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